Monoidal Jantzen filtrations for quantum affine algebras

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(joint work with David Hernandez)

Let $U_q(\widehat{\mathfrak{g}})$ be the (untwisted) quantum affine algebra associated with a complex simple Lie algebra $\mathfrak{g}$ and a generic quantum parameter $q \in \mathbb{C}^{\times}$. This is a Hopf algebra over $\mathbb{C}$. Its finite-dimensional modules form an interesting abelian monoidal category $\mathscr{C}$. For example, the category $\mathscr{C}$ is neither semisimple as an abelian category, nor braided as a monoidal category. In particular, the tensor product $V \otimes W$ is not isomorphic to its opposite $W \otimes V$ for general simple modules $V, W \in \mathscr{C}$. Nevertheless, their Jordan-Hölder factors coincide up to reordering. In other words, we have $[V \otimes W] = [W \otimes V]$ in the Grothendieck ring $K(\mathscr{C})$ for any simple modules $V, W \in \mathscr{C}$, and hence $K(\mathscr{C})$ is commutative. Indeed, this commutativity follows from the injectivity of the so-called q-character homomorphism $\chi_q \colon K(\mathscr{C}) \to \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^{\times}]$ due to Frenkel-Reshetikhin [3], where I is an index set of the simple roots of $\mathfrak{g}$.

By the classification result due to Chari-Pressley [1], the set of simple modules in $\mathscr{C}$ (modulo isomorphism) is in bijection with the set $\mathcal{M} \subset \mathcal{Y}$ of monomials in the variables $Y_{i,a}$. For each $m \in \mathcal{M}$, the corresponding simple module $L(m)$ is of highest weight m, namely $\chi_q(L(m))$ has m as its highest term. The problem to compute $\chi_q(L(m))$ for all $m \in \mathcal{M}$ is of fundamental importance. At the present moment, a closed formula (like the Weyl character formula) is not known.

One possible strategy is to find an algorithm to compute $\chi_q(L(m))$ recursively, analogous to the Kazhdan-Lusztig algorithm. Let us explain this idea briefly. For each $x \in I \times \mathbb{C}^{\times}$, the q-character of the simple module $V_x \coloneqq L(Y_x)$ (called a fundamental module) can be computed by an algorithm due to Frenkel-Mukhin [2]. For each monomial $m = Y_{x_1} \cdots Y_{x_d} \in \mathcal{M}$, if $(x_1, \ldots, x_d)$ is ordered suitably, the corresponding tensor product of the fundamental modules $M(m) \coloneqq V_{x_1} \otimes \cdots \otimes V_{x_d}$ has a simple head isomorphic to $L(m)$. Moreover, there exists a partial ordering of $\mathcal{M}$ (called the Nakajima partial ordering) such that we have

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m,m'}[L(m')]$$

in $K(\mathscr{C})$. The module $M(m)$ is called a standard module. Since we know $\chi_q(M(m))$, it is enough to compute the multiplicities $P_{m,m'}$. For this purpose, we consider a one-parameter (non-commutative) deformation of $K(\mathscr{C})$, called the quantum Grothendieck ring. It was introduced by Nakajima [6] and by Varagnolo-Vasserot [7] for simply-laced $\mathfrak{g}$, and by Hernandez [5] for general $\mathfrak{g}$. Recently, it was also studied in relation with quantum cluster algebras and derived Hall algebras. The quantum Grothendieck ring $K_t(\mathscr{C})$ is an $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of a quantum torus $\mathcal{Y}_t$ deforming $\mathcal{Y}$, stable under a natural anti-involution $y \mapsto \bar{y}$ of $\mathcal{Y}_t$, and comes with a standard $\mathbb{Z}[t^{\pm 1/2}]$-basis $\{M_t(m)\}_{m \in \mathcal{M}}$. Under the specialization $t \to 1$, $M_t(m)$ goes to $[M(m)]$. We can prove (see [6, 5]) that there exists the canonical basis $\{L_t(m)\}_{m\in\mathcal{M}}$ satisfying $\overline{L_t(m)} = L_t(m)$ and

$$M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t)L_t(m')$$

for some $P_{m,m'}(t) \in t\mathbb{Z}[t]$. This characterization enables us to compute the polynomials $P_{m,m'}(t)$ recursively. When $\mathfrak{g}$ is simply-laced, the following result was obtained by using perverse sheaves on quiver varieties.

Theorem 1 (Nakajima [6], Varagnolo-Vasserot [7])**.** *When $\mathfrak{g}$ is simply-laced, the following properties hold:*

- (P1) *Analog of Kazhdan-Lusztig conjecture: under the specialization $t \to 1$, $L_t(m)$ goes to $[L(m)]$, or equivalently, we have $P_{m,m'}(1) = P_{m,m'}$.*
- (P2) *Positivity: for any $m' < m$, we have $P_{m,m'}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

Later, Hernandez [5] conjectured that these properties hold for general $\mathfrak{g}$. Very recently, we obtained some pieces of evidence of this conjecture.

Theorem 2 (F.-Hernandez-Oh-Oya [4])**.** *The property* (P1) *also holds when $\mathfrak{g}$ is of type B. The property* (P2) *holds for general $\mathfrak{g}$.*

Having these results, we propose the following question.

Question 3. *What is representation-theoretic meaning of $K_t(\mathscr{C})$ or $P_{m,m'}(t)$?*

Here we try to answer this question by introducing "monoidal Jantzen filtrations" for any tensor products of fundamental modules. For any sequence $\underline{x} = (x_1, \dots, x_d)$ of elements of $I \times \mathbb{C}^\times$, let $V_{\underline{x}} := V_{x_1} \otimes \cdots \otimes V_{x_d}$ be the corresponding tensor product, which is not necessarily a standard module. By using R-matrices, we are going to define a $U_q(\widehat{\mathfrak{g}})$-modules filtration

$$V_{\underline{x}} \supset \cdots \supset F_{-1} \supset F_0 \supset F_1 \supset \cdots \tag{1}$$

satisfying $F_{\ll 0} = V_{\underline{x}}$ and $F_{\gg 0} = \{0\}$. Then we define an element $[V_{\underline{x}}]_t$ of the t-deformed Grothendieck group $K(\mathscr{C})_t := K(\mathscr{C}) \otimes \mathbb{Z}[t^{\pm 1/2}]$ by

$$[V_{\underline{x}}]_t := \sum_{n\in\mathbb{Z}} [\mathrm{gr}_n^F V_{\underline{x}}] \otimes t^n.$$

Observe that the sets $\{[L(m)]_t := [L(m)] \otimes 1\}_{m\in\mathcal{M}}$ and $\{[M(m)]_t\}_{m\in\mathcal{M}}$ both form $\mathbb{Z}[t^{\pm 1/2}]$-bases of $K(\mathscr{C})_t$. Then we define a $\mathbb{Z}[t^{\pm 1/2}]$-bilinear map $*\colon K(\mathscr{C})_t \times K(\mathscr{C})_t \to K(\mathscr{C})_t$ by

$$[M(m)]_t * [M(m')]_t := t^{\gamma(m,m')}[M(m) \otimes M(m')]_t,$$

where $\gamma\colon \mathcal{M} \times \mathcal{M} \to \frac{1}{2}\mathbb{Z}$ is a skew-symmetric bilinear form on $\mathcal{M}$ related to the structure constants of the quantum torus $\mathcal{Y}_t$. Also, $K(\mathscr{C})_t$ is endowed with a natural involution $\overline{X \otimes f(t)} = X \otimes f(t^{-1})$. Now we propose the following :

Conjecture 4. *The pair $(K(\mathscr{C})_t, *)$ defines a $\mathbb{Z}[t^{\pm 1/2}]$-algebra with anti-involution, and it is isomorphic to the quantum Grothendieck ring $K_t(\mathscr{C})$ identifying the standard basis $\{M_t(m)\}_{m\in\mathcal{M}}$ with the basis $\{[M(m)]_t\}_{m\in\mathcal{M}}$.*

Remark 5. (1) *The associativity of the map $*$ is unclear from the definition.*
(2) *Conjecture 4 implies the above properties* (P1) *&* (P2).

At the present moment, we have the following theorem as a piece of evidence of Conjecture 4. Our proof also uses perverse sheaves on quiver varieties.

Theorem 6 (F.-Hernandez)**.** *Conjecture 4 is true when $\mathfrak{g}$ is simply-laced.*

In the remaining part, we explain how to construct the filtration (1). Let $\mathcal{O} := \mathbb{C}[\![u]\!] \subset \mathcal{K} := \mathbb{C}(\!(u)\!)$. For each fundamental module V_x $(x \in I \times \mathbb{C}^\times)$, we can define its formal spectral parameter deformation $\widehat{V}_x$ over $\mathcal{O}$, which is a $U_q(\widehat{\mathfrak{g}})_{\mathcal{O}}$-module, in a suitable way so that, after the localization $\mathcal{O} \to \mathcal{K}$, we have a unique $U_q(\widehat{\mathfrak{g}})_{\mathcal{K}}$-linear isomorphism :

$$\left(\widehat{V}_x \otimes_{\mathcal{O}} \widehat{V}_y\right)_{\mathcal{K}} \simeq \left(\widehat{V}_y \otimes_{\mathcal{O}} \widehat{V}_x\right)_{\mathcal{K}} \tag{2}$$

respecting highest weight vectors, for any $x, y \in I \times \mathbb{C}^\times$. The isomorphism (2) is called the normalized R-matrix. For a sequence $\underline{x} = (x_1, \dots, x_d)$, we consider the corresponding $\mathcal{O}$-deformed tensor product $\widehat{V}_{\underline{x}} := \widehat{V}_{x_1} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \widehat{V}_{x_d}$. We regard $\widehat{V}_x$ as an $\mathcal{O}$-lattice of its localization $(\widehat{V}_{\underline{x}})_{\mathcal{K}}$. For a fixed $\underline{x}$, let $\underline{x}_0$ denote another sequence obtained from $\underline{x}$ by reordering so that $V_{\underline{x}_0}$ is a standard module, and let $\underline{x}'_0$ denote the sequence opposite to $\underline{x}_0$. Then we have the $U_q(\widehat{\mathfrak{g}})_{\mathcal{K}}$-linear isomorphisms $R\colon (\widehat{V}_{\underline{x}_0})_{\mathcal{K}} \to (\widehat{V}_{\underline{x}})_{\mathcal{K}}$ and $R'\colon (\widehat{V}_{\underline{x}'_0})_{\mathcal{K}} \to (\widehat{V}_{\underline{x}})_{\mathcal{K}}$ by composing the normalized R-matrices. Finally, for each $n \in \mathbb{Z}$, we define

$$F_n := \pi\left(\widehat{V}_{\underline{x}} \cap \sum_{k,l \in \mathbb{Z}; k+l=n} R(u^k \widehat{V}_{\underline{x}_0}) \cap R'(u^l \widehat{V}_{\underline{x}'_0})\right),$$

where $\pi\colon \widehat{V}_{\underline{x}} \to V_{\underline{x}}$ is the specialization $u \to 0$.

References

[1] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups.* Cambridge University Press, Cambridge, 1994.

[2] Edward Frenkel and Evgeny Mukhin. Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras. *Comm. Math. Phys.*, 216(1):23–57, 2001.

[3] Edward Frenkel and Nicolai Reshetikhin. The q-characters of representations of quantum affine algebras and deformations of $\mathscr{W}$-algebras. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 163–205. Amer. Math. Soc., Providence, RI, 1999.

[4] Ryo Fujita, David Hernandez, Se-jin Oh, and Hironori Oya. Isomorphisms among quantum Grothendieck rings and propagation of positivity. Preprint, arXiv:2101.07489. (to appear in *J. Reine Angew. Math.*)

[5] David Hernandez. Algebraic approach to q,t-characters. *Adv. Math.*, 187(1):1–52, 2004.

[6] Hiraku Nakajima. Quiver varieties and t-analogs of q-characters of quantum affine algebras. *Ann. of Math. (2)*, 160(3):1057–1097, 2004.

[7] Michela Varagnolo and Eric Vasserot. Perverse sheaves and quantum Grothendieck rings. In *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, volume 210 of *Progr. Math.*, pages 345–365. Birkhäuser Boston, Boston, MA, 2003.