Abstracts

Deformed Cartan matrices and generalized preprojective algebras of finite type

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(joint work with Kota Murakami)

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. In order to define the deformed $\mathcal{W}$-algebra associated with $\mathfrak{g}$, E. Frenkel and Reshetikhin [2] introduced a two parameters deformation $C(q,t)$ of the Cartan matrix $C = (c_{ij})_{i,j\in I}$ of $\mathfrak{g}$. Letting $D = \mathrm{diag}(d_i \mid i \in I)$ be the minimal left symmetrizer of C, its (i,j)-entry is defined by

$$C_{ij}(q,t) = \begin{cases} q^{d_i}t^{-1} + q^{-d_i}t & \text{if } i = j, \\ [c_{ij}]_q & \text{if } i \neq j. \end{cases}$$

for each $i, j \in I$, where $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$. In the specialization $(q,t) = (1,1)$, it certainly recovers the Cartan matrix C.

Its specialization $C(q) = C(q,1)$ at $t = 1$ is sometimes called the q-Cartan matrix, or the quantum Cartan matrix. It plays an important role in the representation theory of affine quantum groups (e.g. in the description of the diagonal part of the universal R-matrix of the Yangian/quantum loop algebra). Consider the inverse $\widetilde{C}(q) := C(q)^{-1}$ and let $\widetilde{C}_{ij}(q) = \sum_{u\geq 0} \widetilde{c}_{ij}(u)q^u$ denote the Taylor expansion at $q = 0$ of its (i,j)-entry. These Taylor coefficients satisfy some interesting properties: for any $i, j \in I$, we have

(P1) periodicity: $\widetilde{c}_{ij}(u + rh^\vee) = -\widetilde{c}_{ij^*}(u)$ for $u \geq 0$,
(P2) positivity: $\widetilde{c}_{ij}(u) \geq 0$ for $0 \leq u \leq rh^\vee$,
(P3) palindromicity: $\widetilde{c}_{ij}(rh^\vee - u) = \widetilde{c}_{ij^*}(u)$ for $0 \leq u \leq rh^\vee$,

where $r := \max(d_i \mid i \in I)$, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, and $i \mapsto i^*$ is the involution of I induced by the longest element of the Weyl group.

We give a representation-theoretic interpretation for these properties in terms of the generalized preprojective algebra Π. It is introduced by Geiß, Leclerc, and Schröer [5] to generalize the ordinary preprojective algebra associated with the simply-laced Dynkin diagrams to the case of non-simply-laced Dynkin diagrams. The algebra Π is a finite-dimensional self-injective algebra (over a field) defined as a quotient of the Jacobian algebra $\widetilde{\Pi}$ associated with a certain quiver with potential. The algebra $\widetilde{\Pi}$ also appeared in the context of theoretical physics [1] and in the representation theory of the quantum loop algebras [6].

We can equip the algebras Π and $\widetilde{\Pi}$ with a $\mathbb{Z}$-grading following [6]. Associated with each vertex $i \in I$, we have the simple Π-module S_i. In addition, there is the maximal indecomposable iterated self-extension E_i of S_i in the category of graded Π-modules. We can show that the graded Euler-Poincaré paring $\langle E_i, S_j\rangle_q$ is well-defined as a formal Laurent series in q and it can be expressed in terms of the q-Cartan matrix $C(q)$. As its dual statement, we have the following result. Let $\bar{I}_i$

be the graded submodule of the i-th injective Π-module satisfying $\langle E_j, \bar{I}_i \rangle_q = \delta_{ij}$, and $\dim_q \bar{I}_i \in \mathbb{Z}_{\geq 0}[q^{\pm 1}]$ its graded dimension.

Theorem 1 ([3, Theorem A])**.** *For any $i, j \in I$, we have*

$$\widetilde{C}_{ij}(q) = \frac{q^{d_j}}{1 - q^{2rh^\vee}} \left(\dim_q e_i \bar{I}_j - q^{rh^\vee} \dim_q e_i \bar{I}_{j^*} \right),$$

$$\dim_q e_i \bar{I}_j = q^{-d_j} \sum_{u=0}^{rh^\vee} \widetilde{c}_{ij}(u) q^u.$$

This result gives a simple explanation for the above properties (P1) and (P2). Moreover, it enables us to understand the other property (P3) as an incarnation of the self-injectivity of the algebra Π.

As an application, we can compute the graded dimensions of the first extension groups between the generic kernels introduced by Hernandez and Leclerc in [6]. The generic kernels are certain graded $\widetilde{\Pi}$-modules, which can be seen as the additive counterparts of the Kirillov-Reshetikhin (KR) modules over the quantum loop algebra $U_q(L\mathfrak{g})$. More precisely, for each KR module V, the cluster character of the corresponding generic kernel $K_V \in \widetilde{\Pi}$-gmod coincides with the q-character of V after a monomial transformation. Comparing our computations of Ext^1 with the computations of the denominators of the normalized R-matrices between the KR modules due to Oh and Scrimshaw [7] (see also [4]), we are led to the following conjecture. Let $\mathfrak{o}(V, W)$ denote the pole order of the normalized R-matrix $R_{V,W}(z)$ at $z = 1$ for simple $U_q(L\mathfrak{g})$-modules V and W.

Conjecture 2 ([3, Conjecture B])**.** *For any KR modules V and W, we have*

$$\mathfrak{o}(V, W) = \dim \mathrm{Ext}^1_{\widetilde{\Pi}}(K_V, K_W).$$

At this moment, we can check that this conjecture is true as long as the left hand side is known.

References

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